An example related to the N-Bakry-Émery Ricci curvature and a punctured torus.

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This is a note about relates to the N-Bakry-Émery Ricci curvature and the punctured torus. Specifically, we attempt to answer the question, "Can a punctured torus admit metrics which satisfy nonnegative ∞-Bakry-Émery Ricci curvature?"

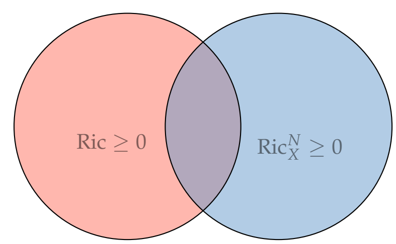
What am I talking about?

The main question that I try to answer in this note is the following:

Question 0.1. *Can a punctured torus admit metrics which satisfy* $\mathrm{Ric}_\phi^\infty \geq 0$*?*

I will give a partial answer to this question later in this note.

I think of the N-Bakry-Émery Ricci curvature is a generalization of Ricci curvature in some sense (we'll get to this later). I see manifolds with non-negative N-Bakry-Émery Ricci curvature ($\mathrm{Ric}_X^N \geq 0$) as comparable to nonnegative Ricci curvature ($\mathrm{Ric} \geq 0$) in the sense that it's expected when manifolds fit in both categories and it's interesting when they fit into the former category but not the latter.



The proposition given later in this note tells us that the punctured torus with non-negative ∞-Bakry-Émery Ricci curvature cannot have a warped product splitting given some conditions. Why did I choose the punctured torus? How is the warped product splitting related? Where do these conditions come from? We will get into this later in the note. But first, we give some much needed definitions.

What definitions do we need?

The most important definition is the N-Bakry-Émery Ricci curvature.

We define the N-Bakry-Émery Ricci tensors as follows:

Definition 0.2. *Let X be a vector field on (M^n, g), a Riemannian manifold. The N-Bakry-Émery tensor is*

$$\mathrm{Ric}_X^N := \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{N-n} X^* \otimes X^*$$

where $\mathcal{L}_X g$ is the Lie derivative of g with respect to X, defined as follows:

$$\begin{aligned} \mathcal{L}_X g : T_pM \times T_pM &\to \mathbb{R} \\ (Y, Z) &\mapsto \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle \end{aligned}$$

and

$$X^* : T_pM \to \mathbb{R}$$
$$Y \mapsto g(X, Y).$$

If $X = \nabla\phi$ where $\phi : M \to \mathbb{R}$ is a smooth function, the N-Bakry-Émery Ricci tensor is

$$\mathrm{Ric}_\phi^N := \mathrm{Ric} + \mathrm{Hess}\,\phi - \frac{1}{N-n} d\phi \otimes d\phi.$$

If $X = \nabla\phi$ and $N = \infty$, then we denote

$$\mathrm{Ric}_\phi := \mathrm{Ric}_\phi^\infty = \mathrm{Ric} + \mathrm{Hess}\,\phi. \tag{1}$$

Remark 0.3. *Note that Ric_X^N is a generalization of Ric_ϕ^N because if $X = \nabla\phi$, then $\mathrm{Ric}_X^N = \mathrm{Ric}_\phi^N$. Similarly, we call Ric_ϕ^N a generalization of* Ric *because if ϕ is constant, then $\mathrm{Ric}_\phi^N = \mathrm{Ric}$.* [1]

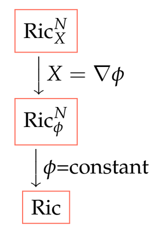
Our next vocabulary term is the notion of a loop being homotopic to another loop.

Definition 0.4. *Given a ray γ and a loop $C : [0, L] \to M$ based at $\gamma(0)$, we say that a loop $\widetilde{C} : [0, L] \to M$ is homotopic to C along γ if there exists $r > 0$ with $\widetilde{C}(0) = \widetilde{C}(L) = \gamma(r)$ and the loop, constructed by joining γ from 0 to r with C from 0 to L and then with γ from r to 0 is homotopic to C, in $\pi_1(M, \gamma(0))$.*

Next, we define the purely topological property (as opposed to having some geometric-ness to it), the geodesic loops to infinity property.

Definition 0.5. *An element $h \in \pi_1(M, \gamma(0))$ has the geodesic loops to infinity property along γ if for any $A \subset M$ compact, there exists a loop $\widetilde{C} \subset M \setminus A$ which is homotopic to a representative loop, C of h along γ.*

[1]



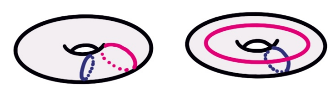


Figure 1: In the figure on the left, the two black loops are homotopic to each other. In the figure on the right, the black loop is not homotopic to the gray loop.

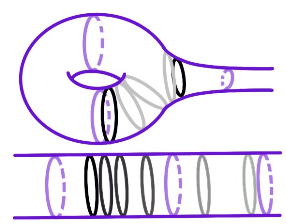


Figure 2: In the figure above, we have a punctured torus, which does not have the geodesic loops to infinity property. The geodesic loop in black on the left gets "stuck" and cannot reach the geodesic loop in black on the right. In the figure below, we have a cylinder, which does have the geodesic loops to infinity property. We see that the black loop is able to homotope to any of the gray loops.

Our last vocabulary term is the warped product splitting.

Definition 0.6. *(M, g) has a warped product splitting if M is diffeomorphic to $\mathbb{R} \times L$ where L is an $(n-1)$-dimensional manifold and there exists $u : \mathbb{R} \to \mathbb{R}^+$ such that $g = dr^2 + u^2(r)g_0$ for a fixed metric g_0. We call g a warped product over $\mathbb{R}$ and we call $u(r)$ the warping function.*

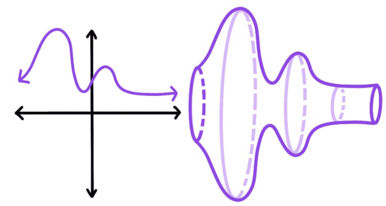


Figure 3: The figure on the left depicts a function, $u(r)$ as in Definition 0.6. The figure on the right depicts a manifold which has a warped product splitting with warping function $u(r)$.

Intuition

The following is Theorem 1.9 from my paper[2].

Theorem 0.7. *Let M^n be complete and noncompact.*

1. *If $\mathrm{Ric}_X^N > 0$ for $N > n$, then $H_{n-1}(M, \mathbb{Z}) = 0$.*
2. *If $\mathrm{Ric}_\phi^N > 0$ for $N \leq 1$ with $\phi < K$ for some $K \in \mathbb{R}$, then $H_{n-1}(M, \mathbb{Z}) = 0$.*
3. *If $\mathrm{Ric}_\phi^\infty > 0$ with $\nabla\phi \to 0$ at ∞, then $H_{n-1}(M, \mathbb{Z}) = 0$.*

This is an interesting question because we know that $\mathrm{Ric}_\phi^\infty \geq A > 0$ cannot occur by [3]. We also know by Theorem 0.7(3) that ϕ must be unbounded. Note in Section 2, we assumed bounds on ϕ, so the topological implications of the Splitting Theorem don't hold for the punctured torus. My conjecture is that there do not exist metrics which satisfy $\mathrm{Ric}_\phi^\infty \geq 0$ on the punctured torus.

It is natural to think that if the punctured torus did admit a metric which satisfies $\mathrm{Ric}_\phi^\infty \geq 0$ that the metric would be a warped product near ∞. I have made progress in proving that the punctured torus cannot satisfy $\mathrm{Ric}_\phi^\infty \geq 0$ if the metric has a warped product splitting near infinity by using theorems in [4]. I hope to develop these ideas to find a class of manifolds which do not admit $\mathrm{Ric}_\phi^\infty \geq 0$. In fact, it is an open question whether there exist non-compact topologically finite surfaces with complete metrics which admit $\mathrm{Ric}_\phi^\infty \geq 0$.

[2] Alice Lim. The splitting theorem and topology of noncompact spaces with nonnegative N-Bakry Émery Ricci curvature. *To appear in the Proceedings of the AMS*, 2020. DOI: https://doi.org/10.1090/proc/15240

[3] William Wylie. A warped product version of the Cheeger-Gromoll splitting theorem. *Transactions of the American Mathematical Society*, 369(9):6661—-6681, 2017

[4] Alice Lim. The splitting theorem and topology of noncompact spaces with nonnegative N-Bakry Émery Ricci curvature. *To appear in the Proceedings of the AMS*, 2020. DOI: https://doi.org/10.1090/proc/15240

Theorem and Proof.

In this section, we provide our theorem and proof.

Theorem 0.8. *Let M be a punctured torus. Let $\mathrm{Ric}_\phi^\infty \geq 0$ and let there exist a point $p \in M$ where $(\mathrm{Ric}_\phi^\infty)_p > 0$. Then M cannot have a warped product splitting, $g = dr^2 + \rho^2(r)d\theta^2$ near infinity if $\lim_{i\to\infty} |\rho'(r_i)|$ exists where $\lim_{i\to\infty} \phi((r_i, \theta_i)) = \infty$.*

Proof.

Let $g = dr^2 + \rho^2(r)d\theta^2$ near infinity as in Figure 4. Let $\rho'(0) < 0$ and $\rho(0) > 0$. Then $\mathrm{Ric}_\phi^\infty(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}) = -\rho\rho'' + \phi'\rho'\rho$ on the punctured part of the punctured torus.

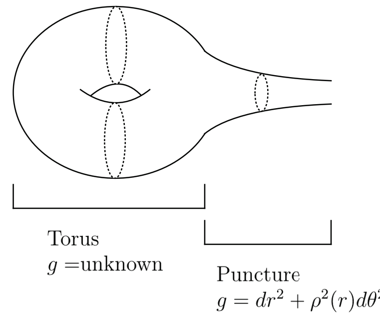


Figure 4: Punctured Torus

Since the punctured torus doesn't have the loops to infinity property, by Theorem 1.9[5], there exists a sequence of points $\{p_i\} \in M$ such that $\lim_{i\to\infty} \phi(p_i) = \infty$. Here, $p_i = (r_i, \theta_i)$ where $\lim_{i\to\infty} r_i = \infty$.

[5] Alice Lim. The splitting theorem and topology of noncompact spaces with nonnegative N-Bakry Émery Ricci curvature. *To appear in the Proceedings of the AMS*, 2020. DOI: https://doi.org/10.1090/proc/15240

Since $\mathrm{Ric}_\phi^\infty(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}) = -\rho e^\phi(\rho' e^{-\phi})'$, Ric_ϕ^∞ is nonnegative if and only if $\rho' e^{-\phi}$ is decreasing everywhere.

We aim to show that $\rho'(r)$ must be non-positive for all $r > 0$. Since $\rho' e^{-\phi}$ is decreasing everywhere, for all $x < y$, we must have $\rho'(x)e^{-\phi(x)} > \rho'(y)e^{-\phi(y)}$. We have that $\rho'(0)e^{-\phi(0)}$ is negative. Suppose there exists a point z such that $\rho'(z)$ is positive. Then $\rho'(z)e^{-\phi(x)}$ is positive, which is a contradiction since $\rho'(0)e^{-\phi(0)} < \rho'(z)e^{-\phi(z)}$. Thus, $\rho'(r)$ is non-positive for all $r > 0$.

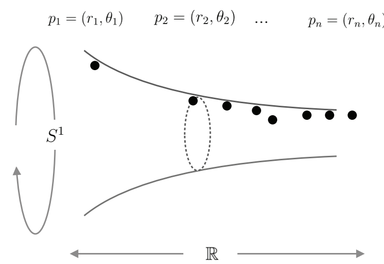


Figure 5: Exists $r_i \to \infty$ such that $\phi(p_i) \to \infty$

Now we have that $\mathrm{Ric}_\phi^\infty(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta})$ is non-negative if and only if $\phi' \leq \frac{\rho''}{\rho}$ for all $r > 0$. Then, integrating both sides, we get $\phi(r) \leq \ln|\rho'(r)|$. Then, plugging in the sequence of points $\{r_i\}$ and letting i go to infinity, we get $\infty = \lim_{i\to\infty} \phi(r_i) \leq \lim_{i\to\infty} \ln|\rho'(r_i)|$.

We must have that $\lim_{i\to\infty} \rho'(r_i)$ is $\pm\infty$. Since $\rho'(r)$ is negative everywhere, $\lim_{i\to\infty} \rho'(r_i)$ can't be ∞, and if $\lim_{i\to\infty} \rho'(r_i)$ is $-\infty$, then we also have a contradiction since ρ is continuous and positive everywhere. □

How could this result be improved?

If we could prove that a punctured torus with a warped product splitting must satisfy $\lim_{i\to\infty} |\rho'(r_i)|$ exists where $\lim_{i\to\infty} \phi((r_i, \theta_i)) = \infty$, then we have a nice succinct theorem about the punctured torus and the Bakry-Émery Ricci curvature. I am hopeful that this punctured torus example will extend to a class of manifolds which satisfy $\mathrm{Ric}_\phi^\infty \geq 0$ and don't admit warped product metrics on their ends. It seems reasonable that if the manifold doesn't satisfy the loops to infinity property and has a hole in it, then the space will not satisfy $\mathrm{Ric}_\phi^\infty = Ag$ with $A \geq 0$ or $A > 0$ with the warped product metric.

References

Alice Lim. The splitting theorem and topology of noncompact spaces with nonnegative N-Bakry Émery Ricci curvature. *To appear in the Proceedings of the AMS*, 2020. DOI: https://doi.org/10.1090/proc/15240.

William Wylie. A warped product version of the Cheeger-Gromoll splitting theorem. *Transactions of the American Mathematical Society*, 369(9):6661—-6681, 2017.